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# Differentiability and Convexity of Fuzzy Mappings

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**Abstract**—Goetschel and Voxman [1] have introduced the notion of a derivative for fuzzy mappings of one variable in a manner different from the usual one. In this paper, we define a differentiable fuzzy mapping of several variables in ways that parallel the definition, proposed by Goetschel and Voxman [1], for a fuzzy mapping of one variable, and then study some basic differentiability properties of fuzzy mappings from the standpoint of convex analysis. 

**Keywords**—Fuzzy numbers, Analysis, Differential, Fuzzy mappings, Convex analysis.

## 1. INTRODUCTION

As is well known, the space of fuzzy numbers, that is, the space of all fuzzy sets $u : R^1 \to [0,1]$ of the real line $R^1$ which are normal, fuzzy convex, upper semicontinuous, and with bounded supports, does not constitute any linear space under operations such as addition, subtraction, and scalar multiplication derived from the usual extension principle. Although a theorem due to Rådström showing that the space of fuzzy numbers embeds as a cone in a Banach space is of theoretical interest, it is not easy to apply. Goetschel and Voxman [1] considered fuzzy numbers from a somewhat different perspective. Basically, they viewed fuzzy numbers in a topological vector space $(\mathcal{V}, d)$, where $d$ is the metric defined on the real vector space $\mathcal{V}$. Using the customary vector space operations together with the metric $d$ on $\mathcal{V}$, they defined differentiation of fuzzy mappings of one variable in ways paralleling the definition for real-valued functions.

The aim of this paper is to extend the notion of differentiability from fuzzy mappings of one variable to fuzzy mappings of several variables. Based on this concept, we study new concepts of pseudoconvexity, invexity, and pseudoinvexity for fuzzy mappings of several variables that are defined in ways paralleling the corresponding definitions for real-valued functions. Our results are motivated by [1–3].

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## 2. PRELIMINARIES

The family of fuzzy numbers will be denoted by $\mathcal{F}_0$. Since each $r \in R^1$ can be considered as a fuzzy number $\tilde{r}$ defined by

$$\tilde{r}(t) = \begin{cases} 1, & \text{if } t = r, \\ 0, & \text{if } t \neq r, \end{cases}$$

it follows that $R^1$ can be embedded in $\mathcal{F}_0$. A fuzzy number $u : R^1 \to [0,1]$ in $\mathcal{F}_0$ is called nonnegative if $u(t) = 0$ for all $t < 0$. We denote the set of all nonnegative fuzzy numbers by $\mathcal{F}_0{}^*$.

It is obvious that the $\alpha$-level set of a fuzzy number $u$ is a closed and bounded interval

$$[a(\alpha), b(\alpha)] = [u]_\alpha = \begin{cases} \{x \in R^1 \mid u(x) \geq \alpha\}, & \text{if } 0 < \alpha \leq 1, \\ \text{cl}\,(\text{supp}\, u), & \text{if } \alpha = 0, \end{cases}$$

where cl (supp $u$) denotes the closure of the support, $\{x \in R^1 \mid u(x) > 0\}$, of $u$. It is easily verified that a fuzzy set $u : R^1 \to [0,1]$ is a fuzzy number if and only if

(i) $[u]_\alpha$ is a closed and bounded interval for each $\alpha \in [0,1]$, and
(ii) $[u]_1 \neq \emptyset$.

Thus, we can identify a fuzzy number $u$ with the parameterized triples

$$\{(a(\alpha), b(\alpha), \alpha) \mid 0 \leq \alpha \leq 1\}, \tag{2.1}$$

where $a(\alpha)$ and $b(\alpha)$ denote the left- and right-hand endpoints of $[u]_\alpha$, respectively.

For easy reference, we state the following results due to Goetschel and Voxman [1].

THEOREM 2.1. *Suppose that $a : [0,1] \to R^1$ and $b : [0,1] \to R^1$ satisfy the conditions*

(1) *$a$ is a bounded nondecreasing function;*
(2) *$b$ is a bounded nonincreasing function;*
(3) *$a(1) \leq b(1)$;*
(4) *for $0 < \beta \leq 1$, $\lim_{\alpha \to \beta^-} a(\alpha) = a(\beta)$ and $\lim_{\alpha \to \beta^-} b(\alpha) = b(\beta)$;*
(5) *$\lim_{\alpha \to 0^+} a(\alpha) = a(0)$ and $\lim_{\alpha \to 0^+} b(\alpha) = b(0)$.*

*Then $u : R^1 \to [0,1]$ defined by*

$$u(t) = \sup\{\alpha \mid a(\alpha) \leq t \leq b(\alpha)\}$$

*is a fuzzy number with parameterization given by (2.1). Moreover, if $u : R^1 \to [0,1]$ is a fuzzy number with parameterization given by (2.1), then the functions $a$ and $b$ satisfy conditions (1)–(5).*

For fuzzy numbers $u, v \in \mathcal{F}_0$ represented by $\{(a(\alpha), b(\alpha), \alpha) \mid 0 \leq \alpha \leq 1\}$ and $\{(c(\alpha), d(\alpha), \alpha) \mid 0 \leq \alpha \leq 1\}$, respectively, and each real number $r$, we define the addition $u + v$ and 'scalar' multiplication $ru$ as follows:

$$\{(a(\alpha), b(\alpha), \alpha) \mid 0 \leq \alpha \leq 1\} + \{(c(\alpha), d(\alpha), \alpha) \mid 0 \leq \alpha \leq 1\}$$
$$= \{(a(\alpha) + c(\alpha), b(\alpha) + d(\alpha), \alpha) \mid 0 \leq \alpha \leq 1\}$$

and

$$ru = \{(ra(\alpha), rb(\alpha), \alpha) \mid 0 \leq \alpha \leq 1\}.$$

It is known [4] that the addition and the nonnegative scalar multiplication on $\mathcal{F}_0$ defined by the above two equations are equivalent to those derived from the usual extension principle, and that $\mathcal{F}_0$ is closed under these operations. To deal with substraction, following [1], we use the vector 'opposite' of $u$

$$(-1)\,u = \{(-a(\alpha), -b(\alpha), \alpha) \mid 0 \leq \alpha \leq 1\}$$

to represent $-u$. It should be noted that $-u$ is not a fuzzy number. The family of parametric representations of members of $\mathcal{F}_0$ and the parametric representations of their negative scalar multiplications form subsets of the vector space

$$\mathcal{V} = \left\{\{(a(\alpha), b(\alpha), \alpha) \mid 0 \le \alpha \le 1\} \mid a : [0,1] \to R^1 \text{ and } b : [0,1] \to R^1 \text{ are bounded functions}\right\},$$

with addition and scalar multiplication defined levelwise.

We metricize $\mathcal{V}$ by the metric

$$d(\{(a(\alpha), b(\alpha), \alpha) \mid 0 \le \alpha \le 1\}, \{(c(\alpha), d(\alpha), \alpha) \mid 0 \le \alpha \le 1\}) \\ = \sup\{\max\{|a(\alpha) - c(\alpha)|, |b(\alpha) - d(\alpha)|\} \mid 0 \le \alpha \le 1\}.$$

Let $\mathcal{V}^* = \{\{(a(\alpha), b(\alpha), \alpha) \mid 0 \le \alpha \le 1\} \mid a, b : [0,1] \to [0,\infty) \text{ are bounded functions}\}$. Then ${\mathcal{F}_0}^* \subset \mathcal{V}^*$. It is easily verified [3] that $\mathcal{V}^*$ is a closed convex cone in $(\mathcal{V}, d)$.

Suppose that $u, v \in \mathcal{F}_0$ are fuzzy numbers represented by $\{(a(\alpha), b(\alpha), \alpha) \mid 0 \le \alpha \le 1\}$ and $\{(c(\alpha), d(\alpha), \alpha) \mid 0 \le \alpha \le 1\}$, respectively. Define a partial ordering $\preceq$ in $\mathcal{F}_0$ by

$$u \preceq v \quad \text{if and only if } a(\alpha) \le c(\alpha) \text{ and } b(\alpha) \le d(\alpha), \qquad \text{for all } \alpha \in [0,1].$$

We say that $u \prec v$, if $u \preceq v$ and there exists $\alpha_0 \in [0,1]$ such that

$$a(\alpha_0) < c(\alpha_0) \qquad \text{or} \qquad b(\alpha_0) < d(\alpha_0).$$

We see that $u = v$, if $u \preceq v$ and $v \preceq u$. It is often convenient to write $v \succeq u$ (respectively, $v \succ u$) in place of $u \preceq v$ ($u \prec v$).

For $u, v \in \mathcal{F}_0$, it is clear that $u - v \in \mathcal{V}^*$ if and only if $u \succeq v$, and that $u - v \in \mathcal{V}^* \setminus \{\tilde{0}\}$ if and only if $u \succ v$. It is also clear that addition and nonnegative scalar multiplication preserve the order on $\mathcal{F}_0$. It can be seen easily that for $u \in \mathcal{F}_0$, $u$ is nonnegative if and only if $u \succeq \tilde{0}$.

Let $H$ be a vector space. Recall that, by definition, a set $K \subseteq H$ is said to be an invex set w.r.t. a function $\eta : K \times K \to H$ if for all $\mathbf{x}$, $\mathbf{y} \in K$, and $0 \le \lambda \le 1$, $\mathbf{y} + \lambda\eta(\mathbf{x}, \mathbf{y}) \in K$. It can be seen easily that every convex set $C \subseteq H$ is an invex set w.r.t. $\eta(\mathbf{x}, \mathbf{y}) = \mathbf{x} + (-1)\mathbf{y}$.

Throughout our presentation, $H$ is a real vector space; $T \subseteq R^n$ is nonempty, open and convex; and $I$ is a nonempty open interval of $R^1$. Let $S \subseteq H$ be nonempty. A mapping $F : S \to \mathcal{F}_0$ is called a fuzzy mapping.

DEFINITION 2.1. *A fuzzy mapping $F : I \to \mathcal{F}_0$ (where $\mathcal{F}_0$ is considered as a subset of $(\mathcal{V}, d)$) is differentiable at $x_0 \in R^1$ if there exists $F'(x_0) \in \mathcal{V}$ such that*

$$\lim_{h \to 0} \frac{F(x_0 + h) - F(x_0)}{h} = F'(x_0). \tag{2.2}$$

*This limit is taken w.r.t. the metric $d$ defined on $\mathcal{V}$.*

REMARK 2.1. (See [1, Theorems 2.2 and 2.3].) Suppose that $F : I \to \mathcal{F}_0$, and for each $x \in I$, let $F(x)$ be represented parametrically by $\{(a(\alpha, x), b(\alpha, x), \alpha) \mid 0 \le \alpha \le 1\}$.

(1) For $x_0 \in I$, if $F$ is differentiable at $x_0 \in R^1$, then $F'(x_0) \in \mathcal{V}$. Furthermore,

$$F'(x_0) = \{(a_x(\alpha, x_0), b_x(\alpha, x_0), \alpha) \mid 0 \le \alpha \le 1\}, \tag{2.3}$$

where $a_x$ and $b_x$ are the partial derivatives of $a(\alpha, x)$ and $b(\alpha, x)$ with respect to $x$.

(2) If the partial derivatives $a_x$ and $b_x$ are continuous, then $F'(x)$ exists for each $x \in I$.

DEFINITION 2.2. *(See [5,6].) Let $C$ be a nonempty convex subset of $H$. A fuzzy mapping $F : C \to \mathcal{F}_0$ is said to be convex, if for $0 \le \lambda \le 1$ and $\mathbf{x}, \mathbf{y} \in C$,*

$$\lambda F(\mathbf{x}) + (1 - \lambda)F(\mathbf{y}) - F(\lambda\mathbf{x} + (1 - \lambda)\mathbf{y}) \in \mathcal{V}^*.$$

DEFINITION 2.3. *(See [7].) Let $K \ne \emptyset$ be an invex subset of $H$ w.r.t. a mapping $\eta : K \times K \to H$. A fuzzy mapping $F : K \to \mathcal{F}_0$ is said to be preinvex on $K$ (w.r.t. $\eta$), if for $0 \le \lambda \le 1$ and $\mathbf{x}, \mathbf{y} \in K$,*

$$\lambda F(\mathbf{x}) + (1 - \lambda)F(\mathbf{y}) - F(\mathbf{y} + \lambda\eta(\mathbf{x}, \mathbf{y})) \in \mathcal{V}^*.$$

## 3. DIFFERENTIATION

In this section, paralleling the definition for a fuzzy mapping of one variable proposed by Goetschel and Voxman [1], we define a differentiable fuzzy mapping of several variables.

Suppose that $F : I \to \mathcal{F}_0$ is differentiable at $x_0 \in I$. Let $\lambda : R^1 \to \mathcal{V}$ be the linear transformation defined by $\lambda(h) = hF'(x_0)$. Then for $h \in R^1$, from the definitions of the metric $d$ on $\mathcal{V}$, and the addition and scalar multiplication of terms in $\mathcal{V}$, we have

$$d\left(\frac{F(x_0+h)-F(x_0)}{h}, F'(x_0)\right) = \frac{1}{|h|} d\left(F(x_0+h), F(x_0)+\lambda(h)\right).$$

So equation (2.2) is equivalent to

$$\lim_{|h|\to 0} \frac{1}{|h|} d\left(F(x_0+h), F(x_0)+\lambda(h)\right) = 0.$$

It is this view of the derivative that inspires the notion of differentiability in the multivariable case.

DEFINITION 3.1. *Suppose that $F : T \to \mathcal{F}_0$, and let $\mathbf{t}_0 \in T$. $F$ is said to be differentiable at $\mathbf{t}_0$ if there is a linear tranformation $\lambda : R^n \to \mathcal{V}$ such that*

$$\lim_{\|\Delta\mathbf{t}\|\to 0} \frac{1}{\|\Delta\mathbf{t}\|} d\left(F(\mathbf{t}_0+\Delta\mathbf{t}), F(\mathbf{t}_0)+\lambda(\Delta\mathbf{t})\right) = 0. \tag{3.1}$$

*$F$ is a differentiable fuzzy mapping if it is differentiable at each $\mathbf{t} \in T$.*

It is not hard to show [8] that, if such a linear tranformation $\lambda$ exists, it is unique. We call this unique linear transformation the differential of $F$ at $\mathbf{t}_0$ and denote it by $D^{(F)}_{\mathbf{t}_0}$.

Suppose that $F : T \to \mathcal{F}_0$ is differentiable at $\mathbf{t}_0 \in T$, and that for each $\mathbf{t} = (t_1, t_2, \dots, t_n) \in T$, $F(\mathbf{t})$ is represented parametrically by

$$\{(a(\alpha, t_1, t_2, \dots, t_n), b(\alpha, t_1, t_2, \dots, t_n), \alpha) \mid 0 \le \alpha \le 1\}. \tag{3.2}$$

Then, by (2.3), for each $\Delta\mathbf{t} = (\Delta t_1, \Delta t_2, \dots, \Delta t_n) \in R^n$, the parametric representation of $D^{(F)}_{\mathbf{t}_0}(\Delta\mathbf{t})$ is given by

$$D^{(F)}_{\mathbf{t}_0}(\Delta\mathbf{t}) = \left\{\left(\sum_{j=1}^{n} \Delta t_j \frac{\partial}{\partial t_j} a(\alpha, \mathbf{t}_0), \sum_{j=1}^{n} \Delta t_j \frac{\partial}{\partial t_j} b(\alpha, \mathbf{t}_0), \alpha\right) \mid 0 \le \alpha \le 1\right\},$$

where $\frac{\partial}{\partial t_j} a$ and $\frac{\partial}{\partial t_j} b$ are the partial derivatives of $a(\alpha, t_1, t_2, \dots, t_n)$ and $b(\alpha, t_1, t_2, \dots, t_n)$, with respect to $t_j$.

DEFINITION 3.2. *Suppose that $F : T \to \mathcal{F}_0$ is differentiable at $\mathbf{t} \in T$. Then the tangent plane to $F$ at $\mathbf{t}$ is defined by*

$$T^{\mathbf{t}}(\Delta\mathbf{t}) = F(\mathbf{t}) + D^{(F)}_{\mathbf{t}}(\Delta\mathbf{t}), \qquad \text{for each } \Delta\mathbf{t} \in R^n.$$

*Notice that if for each $\mathbf{t} = (t_1, t_2, \dots, t_n) \in T$, $F(\mathbf{t})$ is represented parametrically by (3.2), then $T^{\mathbf{t}}(\Delta\mathbf{t})$ is represented parametrically by*

$$\begin{aligned} T^{\mathbf{t}}(\Delta\mathbf{t}) = \Bigg\{\Bigg( & a(\alpha, \mathbf{t}) + \sum_{j=1}^{n} \Delta t_j \frac{\partial}{\partial t_j} a(\alpha, \mathbf{t}), \\ & b(\alpha, \mathbf{t}) + \sum_{j=1}^{n} \Delta t_j \frac{\partial}{\partial t_j} b(\alpha, \mathbf{t}), \alpha\Bigg) \mid 0 \le \alpha \le 1\Bigg\}, \end{aligned} \tag{3.3}$$

*for each $\Delta\mathbf{t} = (\Delta t_1, \Delta t_2, \dots, \Delta t_n) \in R^n$.*

We wish to place a bound on $\{|\Delta t_1|, |\Delta t_2|, \dots, |\Delta t_n|\}$ to ensure that $T^{\mathbf{t}}(\Delta\mathbf{t})$ is a fuzzy number. To this end, motivated by [1], we establish the following criterion.

THEOREM 3.1. *(See [1, Theorem 4.3].) With the notation of (3.3) suppose*

(1) *there exists $k > 0$ such that*

$$a_\alpha(\alpha, \mathbf{t}_0) \geq k, \quad b_\alpha(\alpha, \mathbf{t}_0) \leq -k, \qquad 0 \leq \alpha \leq 1;$$

(2) *there exists $c > 0$ such that for each $j$*

$$\left|a_{t_j\alpha}(\alpha, \mathbf{t}_0)\right| < c, \quad \left|b_{t_j\alpha}(\alpha, \mathbf{t}_0)\right| < c, \qquad 0 \leq \alpha \leq 1;$$

*and either*

(3) *$a(1, \mathbf{t}_0) = b(1, \mathbf{t}_0)$ and $a_{t_j}(1, \mathbf{t}_0) = b_{t_j}(1, \mathbf{t}_0)$ for each $j$, or*

(4) *$b(1, \mathbf{t}_0) - a(1, \mathbf{t}_0) = d > 0$.*

*If (3) holds and $|\Delta t_j| \leq k/nc$ for each $j$, or if (4) holds and $|\Delta t_j| \leq \min\{k/nc, d/2nl\}$, where $l = \max\{|a_{t_j}(1, \mathbf{t}_0)|, |b_{t_j}(1, \mathbf{t}_0)| \mid j = 1, \ldots, n\}$, then $T^{\mathbf{t}_0}(\Delta\mathbf{t})$ is a fuzzy number.*

PROOF. It suffices to show that $T^{\mathbf{t}_0}(\Delta\mathbf{t})$ satisfies the five conditions of Theorem 2.1.

Suppose that $0 \leq \alpha \leq \beta \leq 1$. Then $n + 1$ applications of the mean value theorem yield real numbers $\gamma_0, \gamma_1, \ldots, \gamma_n$ such that

$$\begin{aligned}
&a(\beta, \mathbf{t}_0) + \sum_{j=1}^{n} \Delta t_j \frac{\partial}{\partial t_j} a(\beta, \mathbf{t}_0) - \left(a(\alpha, \mathbf{t}_0) + \sum_{j=1}^{n} \Delta t_j \frac{\partial}{\partial t_j} a(\alpha, \mathbf{t}_0)\right) \\
&\quad = a(\beta, \mathbf{t}_0) - a(\alpha, \mathbf{t}_0) + \sum_{j=1}^{n} \Delta t_j \left(\frac{\partial}{\partial t_j} a(\beta, \mathbf{t}_0) - \frac{\partial}{\partial t_j} a(\alpha, \mathbf{t}_0)\right) \\
&\quad = \left(a_\alpha(\gamma_0, \mathbf{t}_0) + \sum_{j=1}^{n} \Delta t_j a_{t_j\alpha}(\gamma_j, \mathbf{t}_0)\right)(\beta - \alpha) \\
&\quad \geq (k - c(|\Delta t_1| + \cdots + |\Delta t_n|))(\beta - \alpha) \geq 0.
\end{aligned}$$

In addition, since $F(\mathbf{t}_0)$ is a fuzzy number, and since the functions

$$\frac{\partial}{\partial t_j} a(\cdot, \mathbf{t}_0) : [0, 1] \to R^1 \qquad \text{and} \qquad \frac{\partial}{\partial t_j} b(\cdot, \mathbf{t}_0) : [0, 1] \to R^1$$

are continuous ($a_{t_j\alpha}$ and $b_{t_j\alpha}$ exist by (2) of Theorem 3.1), it follows that

$$\left\{a(\alpha, \mathbf{t}) + \sum_{j=1}^{n} \Delta t_j \frac{\partial}{\partial t_j} a(\alpha, \mathbf{t}) \mid 0 \leq \alpha \leq 1\right\}$$

is bounded. Thus, (1) of Theorem 2.1 is established. Condition (2) of this theorem is established in an analogous fashion. Condition (3) of Theorem 2.1 obviously holds if Part (3) of Theorem 3.1 is satisfied. Moreover, if (4) of this theorem is satisfied, then to see (3) of Theorem 2.1 holds, suppose that $b(1, \mathbf{t}_0) - a(1, \mathbf{t}_0) = d > 0$ and $|\Delta t_j| \leq \min\{k/nc, d/2nl\}$ where $l = \max\{|a_{t_j}(1, \mathbf{t}_0)|, |b_{t_j}(1, \mathbf{t}_0)|,\ j = 1, \ldots, n\}$. Then

$$\begin{aligned}
&b(1, \mathbf{t}_0) + \sum_{j=1}^{n} \Delta t_j \frac{\partial}{\partial t_j} b(1, \mathbf{t}_0) - \left(a(1, \mathbf{t}_0) + \sum_{j=1}^{n} \Delta t_j \frac{\partial}{\partial t_j} a(1, \mathbf{t}_0)\right) \\
&\quad = b(1, \mathbf{t}_0) - a(1, \mathbf{t}_0) + \sum_{j=1}^{n} \Delta t_j \left(\frac{\partial}{\partial t_j} b(1, \mathbf{t}_0) - \frac{\partial}{\partial t_j} a(1, \mathbf{t}_0)\right) \\
&\quad \geq d - 2l(|\Delta t_1| + \cdots + |\Delta t_n|) \geq 0.
\end{aligned}$$

Finally, since $F(\mathbf{t}_0)$ is a fuzzy number, and since the functions

$$\frac{\partial}{\partial t_j}a(\cdot,\mathbf{t}_0) : [0,1] \to R^1 \qquad \text{and} \qquad \frac{\partial}{\partial t_j}b(\cdot,\mathbf{t}_0) : [0,1] \to R^1$$

are continuous, it follows that $T^{\mathbf{t}_0}(\Delta\mathbf{t})$ satisfies the conditions (4) and (5) of Theorem 2.1 and therefore is a fuzzy number. This completes the proof.

Suppose that $F : T \to \mathcal{F}_0$ is differentiable at $\mathbf{t}_0 \in T$. Let $v_1, v_2, \dots, v_n \in \mathcal{V}$. For each $\Delta\mathbf{t} = (\Delta t_1, \Delta t_2, \dots, \Delta t_n) \in R^n$, denote the term $\Delta t_1 v_1 + \Delta t_2 v_2 + \cdots + \Delta t_n v_n$ by $\Delta\mathbf{t}\cdot\mathbf{v}$. Then we can show that there exists a $\delta > 0$ such that if

$$\max\{|\Delta t_1|, |\Delta t_2|, \dots, |\Delta t_n|\} < \delta,$$

then $d(F(\mathbf{t}_0 + \Delta\mathbf{t}), F(\mathbf{t}_0) + \Delta\mathbf{t}\cdot\mathbf{v}) \geq d(F(\mathbf{t}_0+\Delta\mathbf{t}), D_{\mathbf{t}_0}^{(F)}(\Delta\mathbf{t}))$. To see this, by contradiction, if no such $\delta$ exists, then $\Delta\mathbf{t}\cdot\mathbf{v} \neq D_{\mathbf{t}_0}^{(F)}(\Delta\mathbf{t})$ and there is a sequence

$$\{\boldsymbol{\Delta}_k = (\Delta_{1,k}, \Delta_{2,k}, \dots, \Delta_{n,k})\},$$

in $R^n$ which converges to the zero vector $(0, 0, \dots, 0)$, such that for each $k$,

$$d\left(F\left(\mathbf{t}_0 + \boldsymbol{\Delta}_k\right), F\left(\mathbf{t}_0\right) + \boldsymbol{\Delta}_k\cdot\mathbf{v}\right) < d\left(F\left(\mathbf{t}_0 + \boldsymbol{\Delta}_k\right), F\left(\mathbf{t}_0\right) + D_{\mathbf{t}_0}^{(F)}\left(\boldsymbol{\Delta}_k\right)\right).$$

Since $F$ is differentiable at $\mathbf{t}_0$, it follows that

$$\lim_{k\to\infty} \frac{1}{\|\boldsymbol{\Delta}_k\|} d\left(F\left(\mathbf{t}_0 + \boldsymbol{\Delta}_k\right), F\left(\mathbf{t}_0\right) + \boldsymbol{\Delta}_k\cdot\mathbf{v}\right) = 0,$$

which implies that $\Delta\mathbf{t}\cdot\mathbf{v} = D_{\mathbf{t}_0}^{(F)}(\Delta\mathbf{t})$. This contradicts our assumption; therefore, the desired $\delta$ must exist.

From Theorem 3.1, we immediately obtain the following.

THEOREM 3.2. *Let $\mathbf{t} \in T$. Suppose that $F : T \to \mathcal{F}_0$ is differentiable at $\mathbf{t}$. Then the tangent plane, $T^{\mathbf{t}}(\Delta\mathbf{t})$, to $F$ at $\mathbf{t}$ provides the best local linear approximation of $F$ at $\mathbf{t}$.*

## 4. CONVEXITY OF FUZZY MAPPINGS

Based on the concept of differentiation of fuzzy mappings of several variables, new concepts of pseudoconvexity, invexity, and pseudoinvexity for fuzzy mappings of several variables are defined in ways that parallel the corresponding definitions for real-valued functions.

DEFINITION 4.1. *A differentiable fuzzy mapping $F : T \to \mathcal{F}_0$ is called pseudoconvex if for all $\mathbf{x}$, $\mathbf{y} \in T$,*

$$D_{\mathbf{y}}^{(F)}(\mathbf{x}-\mathbf{y}) \in \mathcal{V}^* \Longrightarrow F(\mathbf{x}) - F(\mathbf{y}) \in \mathcal{V}^*.$$

In view of Definition 4.1, we have the following.

THEOREM 4.1. *Let $F : T \to \mathcal{F}_0$ be pseudoconvex and $\mathbf{x}_0 \in T$. If $\mathbf{x}_0$ satisfies*

$$D_{\mathbf{x}_0}^{(F)}(\mathbf{x}-\mathbf{x}_0) \in \mathcal{V}^*, \qquad \textit{for all } \mathbf{x} \in T,$$

*then $\mathbf{x}_0$ is a point where the mapping $F$ achieves its global minimum on $T$.*

DEFINITION 4.2. *Suppose that $F : T \to \mathcal{F}_0$ is differentiable, and let $\eta : T \times T \to R^n$. $F$ is said to be:*

(1) *invex w.r.t. $\eta$ if for all $\mathbf{x}$, $\mathbf{y} \in T$,*

$$F(\mathbf{x}) - F(\mathbf{y}) - D_{\mathbf{y}}^{(F)}(\eta(\mathbf{x},\mathbf{y})) \in \mathcal{V}^*;$$

(2) *pseudoinvex w.r.t. $\eta$ if for all $\mathbf{x}$, $\mathbf{y} \in T$,*

$$D_{\mathbf{y}}^{(F)}(\eta(\mathbf{x},\mathbf{y})) \in \mathcal{V}^* \Longrightarrow F(\mathbf{x}) - F(\mathbf{y}) \in \mathcal{V}^*.$$

It is clear, by taking $\eta(\mathbf{x},\mathbf{y}) = \mathbf{x} - \mathbf{y}$, that every pseudoconvex fuzzy mapping is pseudoinvex. It is also obvious from the definitions that every invex fuzzy mapping is pseudoinvex w.r.t. the same $\eta$.

From Part (2) of Definition 4.2, we have the following.

THEOREM 4.2. *Suppose that $F : T \to \mathcal{F}_0$ is pseudoinvex w.r.t. a function $\eta : T \times T \to R^n$, and let $\mathbf{x}_0 \in T$. If $\mathbf{x}_0$ satisfies*

$$D_{\mathbf{y}}^{(F)}(\eta(\mathbf{x},\mathbf{y})) \in \mathcal{V}^*, \qquad \text{for all } \mathbf{x} \in T,$$

*then $\mathbf{x}_0$ is a point where the mapping $F$ achieves its global minimum on $T$.*

THEOREM 4.3. *Let $F : T \to \mathcal{F}_0$ be a differentiable convex fuzzy mapping. Then for all $\mathbf{x}$, $\mathbf{y} \in T$,*

$$F(\mathbf{x}) - F(\mathbf{y}) - D_{\mathbf{y}}^{(F)}(\mathbf{x} - \mathbf{y}) \in \mathcal{V}^*. \tag{4.1}$$

PROOF. Let $F : T \to \mathcal{F}_0$ be a differentiable convex fuzzy mapping. Then, by the convexity of $F$, for all $\mathbf{x}$, $\mathbf{y} \in T$ and $\lambda \in (0,1)$,

$$\lambda F(\mathbf{x}) + (1-\lambda) F(\mathbf{y}) - F(\lambda \mathbf{x} + (1-\lambda)\mathbf{y}) \in \mathcal{V}^*. \tag{4.2}$$

Rewrite (4.2) as

$$\lambda (F(\mathbf{x}) - F(\mathbf{y})) - (F(\mathbf{y} + \lambda(\mathbf{x}-\mathbf{y})) - F(\mathbf{y})) \in \mathcal{V}^*. \tag{4.3}$$

From the differentiability of $F$, we have

$$\lim_{\|\lambda(\mathbf{x}-\mathbf{y})\| \to 0} \frac{1}{\|\lambda(\mathbf{x}-\mathbf{y})\|} d\left(F(\mathbf{y} + \lambda(\mathbf{x}-\mathbf{y})) - F(\mathbf{y}), D_{\mathbf{y}}^{(F)}(\lambda(\mathbf{x}-\mathbf{y}))\right) = 0.$$

Since $D_{\mathbf{y}}^{(F)}$ is linear and the metric $d$ is positive homogeneous, it follows that

$$\lim_{\lambda \to 0^+} \frac{1}{\|\mathbf{x}-\mathbf{y}\|} d\left(\frac{F(\mathbf{y} + \lambda(\mathbf{x}-\mathbf{y})) - F(\mathbf{y})}{\lambda}, D_{\mathbf{y}}^{(F)}(\mathbf{x}-\mathbf{y})\right) = 0,$$

which implies that

$$\lim_{\lambda \to 0^+} \frac{F(\mathbf{y} + \lambda(\mathbf{x}-\mathbf{y})) - F(\mathbf{y})}{\lambda} = D_{\mathbf{y}}^{(F)}(\mathbf{x}-\mathbf{y}). \tag{4.4}$$

The limit is taken with respect to the metric $d$. In view of (4.4) and the fact that $\mathcal{V}^*$ is a closed convex cone in $(\mathcal{V}, d)$, dividing (4.3) by $\lambda > 0$ and taking the limit as $\lambda \to 0^+$ gives $F(\mathbf{x}) - F(\mathbf{y}) - D_{\mathbf{y}}^{(F)}(\mathbf{x}-\mathbf{y}) \in \mathcal{V}^*$. This completes the proof.

EXAMPLE 4.1. To illustrate Theorem 4.3 with a specific mapping, let us consider the fuzzy mapping $F : D = \{(t_1, t_2) : t_1 > 0, t_2 > 0\} \to \mathcal{F}_0$ defined parametrically by

$$F(t_1, t_2) = \left\{\left(-(t_1 + t_2)\sqrt{1-\alpha}, \left(t_1^2 + t_2^2\right)\sqrt{1-\alpha}, \alpha\right) \mid 0 \le \alpha \le 1\right\}.$$

It is easily seen that $F$ is a differentiable convex fuzzy mapping. It can be easily checked that for each $\mathbf{x} = (x_1, x_2)$, $\mathbf{y} = (y_1, y_2) \in D$,

$$D_{\mathbf{y}}^{(F)}(\mathbf{x}-\mathbf{y}) = \Big\{\Big(-((x_1 - y_1) + (x_2 - y_2))\sqrt{1-\alpha}, (2(x_1 - y_1) y_1 + 2(x_2 - y_2) y_2)\sqrt{1-\alpha}, \alpha\Big) \mid 0 \le \alpha \le 1\Big\}.$$

It follows that

$$F(\mathbf{x}) - F(\mathbf{y}) - D_{\mathbf{y}}^{(F)}(\mathbf{x}-\mathbf{y}) = \left\{\left(0, \left[(x_1 - y_1)^2 + (x_2 - y_2)^2\right]\sqrt{1-\alpha}, \alpha\right) \mid 0 \le \alpha \le 1\right\},$$

which shows that $F$ satisfies (4.1).

COROLLARY 4.1. *Let $F : T \to \mathcal{F}_0$ be a differentiable convex fuzzy mapping. Then $F$ is pseudoconvex on $T$.*

COROLLARY 4.2. *Let $F : T \to \mathcal{F}_0$ be a differentiable convex fuzzy mapping and $\mathbf{x}_0 \in T$. If $\mathbf{x}_0$ satisfies*

$$D^{(F)}_{\mathbf{x}_0}(\mathbf{x}-\mathbf{x}_0) \in \mathcal{V}^*, \qquad \text{for all } \mathbf{x} \in T,$$

*then $\mathbf{x}_0$ is a point where the mapping $F$ achieves its global minimum on $T$.*

The following theorem gives a sufficient condition for an invex fuzzy mapping to be pseudoconvex.

THEOREM 4.4. *Let $F : T \to \mathcal{F}_0$ be invex w.r.t. a function $\eta : T \times T \to R^n$. If*

$$D^{(F)}_{\mathbf{y}}(\eta(\mathbf{x},\mathbf{y})) - D^{(F)}_{\mathbf{y}}(\mathbf{x}-\mathbf{y}) \in \mathcal{V}^*, \tag{4.5}$$

*for all $\mathbf{x}$, $\mathbf{y} \in T$, then $F$ is pseudoconvex.*

PROOF. Let $F : T \to \mathcal{F}_0$ be invex w.r.t. $\eta$. If $F$ satisfies (4.5), then we have

$$D^{(F)}_{\mathbf{y}}(\eta(\mathbf{x},\mathbf{y})) = D^{(F)}_{\mathbf{y}}(\mathbf{x}-\mathbf{y}) + w, \qquad \text{for some } w \in \mathcal{V}^*.$$

By the invexity of $F$, we have $F(\mathbf{x}) - F(\mathbf{y}) - (D^{(F)}_{\mathbf{y}}(\mathbf{x}-\mathbf{y}) + w) \in \mathcal{V}^*$. Thus, we conclude that $D^{(F)}_{\mathbf{y}}(\mathbf{x}-\mathbf{y}) \in \mathcal{V}^* \Longrightarrow F(\mathbf{x}) - F(\mathbf{y}) \in \mathcal{V}^*$, which completes the proof.

In what follows, suppose that $K \subseteq R^n$ is a nonempty open invex set w.r.t. a function $\eta : K \times K \to R^n$.

Using an argument similar to that used in establishing Theorem 4.3, the following result can be easily proved.

THEOREM 4.5. *If $F : K \to \mathcal{F}_0$ is differentiable with preinvex w.r.t. $\eta$, then $F$ is invex w.r.t. the same $\eta$.*

THEOREM 4.6. *Let $F : K \to \mathcal{F}_0$ be differentiable and preinvex w.r.t. $\eta$. Then $\mathbf{x}_0 \in K$ is a point where the mapping $F$ achieves its global minimum on $T$ if and only if*

$$D^{(F)}_{\mathbf{x}_0}(\eta(\mathbf{x},\mathbf{x}_0)) \in \mathcal{V}^*, \qquad \text{for all } \mathbf{x} \in K. \tag{4.6}$$

PROOF. Let $\mathbf{x}_0$ be a point where the mapping $F$ achieves its global minimum on $T$, and $\mathbf{x} \in K$. Since $K$ is an invex set, then for $0 < t < 1$, we have $\mathbf{x}_t = \mathbf{x}_0 + t\eta(\mathbf{x},\mathbf{x}_0) \in K$, and so

$$F(\mathbf{x}_0 + t\eta(\mathbf{x},\mathbf{x}_0)) - F(\mathbf{x}_0) \in \mathcal{V}^*, \qquad \text{for } 0 < t < 1, \tag{4.7}$$

Since $F$ is differentiable at $\mathbf{x}_0$, i.e.,

$$\lim_{\|t\eta(\mathbf{x},\mathbf{x}_0)\| \to 0} \frac{1}{\|t\eta(\mathbf{x},\mathbf{x}_0)\|} d\left(F(\mathbf{x}_0 + t\eta(\mathbf{x},\mathbf{x}_0)) - F(\mathbf{x}_0), D^{(F)}_{\mathbf{x}_0}(t\eta(\mathbf{x},\mathbf{x}_0))\right) = 0,$$

and since $D^{(F)}_{\mathbf{x}_0}$ is linear and the metric $d$ is positive homogeneous, it follows that

$$\lim_{t \to 0^+} \frac{F(\mathbf{x}_0 + t\eta(\mathbf{x},\mathbf{x}_0))}{t} = D^{(F)}_{\mathbf{x}_0}(\eta(\mathbf{x},\mathbf{x}_0)). \tag{4.8}$$

The limit is taken with respect to the metric $d$. In view of (4.7) and (4.8), we have $F(\mathbf{x}) - F(\mathbf{x}_0) - D^{(F)}_{\mathbf{x}_0}(\eta(\mathbf{x},\mathbf{x}_0)) \in \mathcal{V}^*$. It follows that $D^{(F)}_{\mathbf{x}_0}(\eta(\mathbf{x},\mathbf{x}_0)) \in \mathcal{V}^*$.

Conversely, let $\mathbf{x}_0 \in K$ satisfy (4.6). Since $F$ is a differentiable preinvex fuzzy mapping, it follows from Theorem 4.5 that $F$ is invex. Thus, we have for all $\mathbf{x}$, $\mathbf{y} \in K$,

$$F(\mathbf{x}) - F(\mathbf{y}) - D^{(F)}_{\mathbf{y}}(\eta(\mathbf{x},\mathbf{y})) \in \mathcal{V}^*. \tag{4.9}$$

Now, taking $\mathbf{y} = \mathbf{x}_0$ in (4.9), we have $F(\mathbf{x}) - F(\mathbf{x}_0) - D^{(F)}_{\mathbf{x}_0}(\eta(\mathbf{x},\mathbf{x}_0)) \in \mathcal{V}^*$. By (4.6), we must have for all $\mathbf{x} \in K$,

$$F(\mathbf{x}) - F(\mathbf{x}_0) \in \mathcal{V}^*,$$

which completes the proof.

THEOREM 4.7. *Let $F : T \to \mathcal{F}_0$ be invex w.r.t. a function $\eta : T \times T \to R^n$ and $\mathbf{x}_0 \in T$. If $\mathbf{x}_0$ satisfies*

$$D_{\mathbf{x}_0}^{(F)}\left(\eta\left(\mathbf{x}, \mathbf{x}_0\right)\right) \in \mathcal{V}^*, \qquad \text{for all } \mathbf{x} \in T,$$

*then $\mathbf{x}_0$ is a point where the mapping $F$ achieves its global minimum on $T$.*

## 5. DISCUSSIONS

Motivated by earlier research work, we have proposed in this paper the concept of differentiability of fuzzy mappings on $R^n$ in a manner different from the usual one [9]. Based on this concept, generalized convex fuzzy mappings such as pseudoconvex fuzzy mappings, invex fuzzy mappings and pseudoinvex fuzzy mappings are introduced in ways that closely parallel the corresponding definitions for real-valued functions. We have developed some optimality criteria for differentiable convex fuzzy mappings and differentiable preinvex fuzzy mappings. These results also give rise to optimality and duality results for fuzzy nonlinear programming problems. However, further research is needed to clarify the exact detailed relationship.

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